THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2078 Honours Algebraic Structures 2023-24
Tutorial 9 Problems
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- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. Let I, J be ideals of R, then show that $I \cap J$ and $I + J := \{a + b | \, a \in I, b \in J\}$ are ideals of R.

2. Let $\{I_i\}_{i=1}^{\infty}$ be an increasing sequence of ideals in R, i.e. I_i are ideals of R and $I_i \subset I_{i+1}$. Prove that $\bigcup_{i=1}^{\infty} I_i$ is again an ideal of R.

3. Let $S \subset R$ be a subset of a commutative ring, then denote $\langle S \rangle := \{\sum_{i=1}^{n} a_i s_i | \, a_i \in R, s_i \in S\}$. Prove that $\langle S \rangle$ is an ideal and $\langle S \rangle = \bigcap_{S \subset I} I$ where the intersection is taken over all ideals I of R. Hence, show that if J is some ideal of R such that $S \subset J$, then $\langle S \rangle \subset J$. (Does this look familiar?)

4. Let $I \subset \mathbb{Z}$ be an ideal, suppose that I is nontrivial, prove that $I = n\mathbb{Z}$, where n is the smallest positive integer in I.

5. Let R be a ring, prove that there exists a unique ring homomorphism $\varphi : \mathbb{Z} \to R$. By Q1 the ideal $\ker \varphi$ is equal to $n\mathbb{Z}$ for some positive integer n, this number is called the characteristic of the ring R and is denoted as $\mathrm{Char}(R)$.
   (a) Prove that if R has characteristic $n > 0$, then for any $r \in R$, $n \cdot r = \underbrace{r + ... + r}_{n \text{ times}} = 0$.
   (b) Suppose that R is a ring without zero divisors, prove that $\mathrm{Char}(R)$ is either 0 or a prime number.
   (c) Suppose that R is a commutative ring with $\mathrm{Char}(R) = p$, which is prime. Show that $f : R \to R$ defined by $f(x) = x^p$ is a ring homomorphism. Furthermore, show that it is injective if R is an integral domain. This homomorphism is known as the Frobenius homomorphism.

6. Let $a \in R$ be an element, show that there exists a unique ring homomorphism $\varphi : \mathbb{Z}[x] \to R$ so that $\varphi(x) = a$. Prove that $\mathrm{im}(\varphi)$ is the smallest subring of R containing a, i.e. if $R' \subset R$ is a subring, and $a \in R'$, then $\mathrm{im}(\varphi) \subset R'$. (Guess what happens if $a \in R$ is replaced by a subset $S \subset R$.)

7. Let R be a commutative ring, and $\mathrm{Nil}(R) := \{r \in R | \, r^n = 0 \text{ for some } n \in \mathbb{Z}_{>0}\}$, prove that $\mathrm{Nil}(R)$ is an ideal. This ideal is known as the nilradical of R.